

Secants, bitangents, and their congruences

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Abstract A congruence is a surface in the Grassmannian $\mathrm{Gr}(1, \mathbb{P}^3)$ of lines in projective 3-space. To a space curve C , we associate the Chow hypersurface in $\mathrm{Gr}(1, \mathbb{P}^3)$ consisting of all lines which intersect C . We compute the singular locus of this hypersurface, which contains the congruence of all secants to C . A surface S in \mathbb{P}^3 defines the Hurwitz hypersurface in $\mathrm{Gr}(1, \mathbb{P}^3)$ of all lines which are tangent to S . We show that its singular locus has two components for general enough S : the congruence of bitangents and the congruence of inflectional tangents. We give new proofs for the bidegrees of the secant, bitangent and inflectional congruences, using geometric techniques such as duality, polar loci and projections. We also study the singularities of these congruences. In order to make this article self-contained and accessible to a broader audience, we discuss some of the relevant background material in detail.

1 Introduction

The aim of this article is to study subvarieties of Grassmannians which naturally arise from subvarieties of complex projective 3-space $\mathbb{P}^3 := \mathbb{P}_{\mathbb{C}}^3$. In particular we are mostly interested in threefolds and surfaces in $\mathrm{Gr}(1, \mathbb{P}^3)$, classically called *line complexes* and *congruences*, respectively. We are interested in determining their

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classes in the Chow ring of $\text{Gr}(1, \mathbb{P}^3)$ and their singular loci. In the following, the singular points of a congruence are simply its singular points as a subvariety of the Grassmannian $\text{Gr}(1, \mathbb{P}^3)$, although classically a singular point of a congruence is a point in \mathbb{P}^3 through which there pass infinitely many lines of the congruence. Nowadays, the latter points are called *fundamental points*.

The *Chow complex* $\text{CH}_0(C)$ of a curve $C \subset \mathbb{P}^3$ is the variety of all lines in $\text{Gr}(1, \mathbb{P}^3)$ intersecting C . The *Hurwitz complex* $\text{CH}_1(S)$ of a surface S in \mathbb{P}^3 is the closure of the set of all lines in $\text{Gr}(1, \mathbb{P}^3)$ which are tangent to S at a smooth point. Our main results are the following:

Theorem 1.1. *Let C be a space curve that is not contained in a plane and has degree d , geometric genus g , and ordinary singularities x_1, \dots, x_s with multiplicities r_1, \dots, r_s . Then*

- $\text{Sing}(\text{CH}_0(C)) = \text{Sec}(C) \cup \bigcup_{i=1}^s \{L \in \text{Gr}(1, \mathbb{P}^3) \mid x_i \in L\}$, where $\text{Sec}(C)$ is the locus of secant lines to C (see Theorem 3.7).
- The bidegree of $\text{Sec}(C)$ is

$$\left(\frac{1}{2}(d-1)(d-2) - g - \sum_{i=1}^s \frac{1}{2}r_i(r_i-1), \frac{1}{2}d(d-1) \right)$$

(see Theorem 3.9).

- If C is smooth, then the singular locus of $\text{Sec}(C)$ consists of all lines that intersect C with total multiplicity of at least three (see Propositions 7.1, 7.2 and Theorem 7.4).

Let $S \subset \mathbb{P}^3$ be a general surface of degree $d \geq 4$. Then

- $\text{Sing}(\text{CH}_1(S)) = \text{Bit}(S) \cup \text{Infl}(S)$, where $\text{Bit}(S)$ is the locus of bitangents to S and $\text{Infl}(S)$ is the locus of inflectional tangents to S (see Theorem 4.2).
- The bidegree of $\text{Bit}(S)$ is

$$\left(\frac{1}{2}d(d-1)(d-2)(d-3), \frac{1}{2}d(d-2)(d-3)(d+3) \right)$$

(see Theorem 4.3).

- The bidegree of $\text{Infl}(S)$ is

$$(d(d-1)(d-2), 3d(d-2))$$

(see Theorem 4.3).

- The singular locus of $\text{Infl}(S)$ consists of all lines that are inflectional tangents at at least two points of S or that intersect S with multiplicity at least four at some point (see Theorem 7.5).

The bidegree of $\text{Infl}(S)$ was already calculated in [23, Prop. 4.1]. The bidegrees of $\text{Bit}(S)$, $\text{Infl}(S)$ and $\text{Sec}(C)$ (only for smooth C) were already proven in [2], a paper to which we owe a great debt. However, we give alternative, more geometric proofs,

not relying on Chern class techniques. The singular locus of $\text{Sec}(C)$ was also partly described in [2, Lemma 2.3].

Using duality we can also show relations between some of the above varieties.

Theorem 1.2. *If C is a smooth space curve that is not contained in a plane, then the secant lines of C are dual to the bitangent lines of the dual surface C^\vee and the tangent lines of C are dual to the inflectional tangent lines of C^\vee (see Theorem 5.5).*

Congruences and line complexes have been actively studied both in the nineteenth century and in modern times. The study of congruences goes back to Kummer [21], who classified those of *order one*, where the order of a congruence is the number of lines in the congruence that pass through a general point in 3-space. Specifically Chow complexes of space curves were first introduced by Cayley [4] and later generalized to arbitrary varieties by Chow and van der Waerden [5]. Many classical results from the second half of the 19th century can be found in Jespers's monograph [17]. Hurwitz complexes and further generalizations to general Grassmannians known as *higher associated* or *coisotropic hypersurfaces* are studied in [11, 19, 30]. Catanese [3] showed that the Chow complexes of space curves and Hurwitz complexes of surfaces are exactly the self-dual complexes in the Grassmannian $\text{Gr}(1, \mathbb{P}^3)$. Ran [26] studied surfaces of order one in general Grassmannians and gave a modern proof of Kummer's classification. Congruences play a role in algebraic vision / multi-view geometry, where *cameras* are modeled as maps from \mathbb{P}^3 to congruences [25]. The multidegree of the image of several of those cameras is computed in the article [9] by Escobar and Knutson in this volume.

In Section 2 we recall preliminary facts about Grassmannians and their subvarieties. In Section 3 we study the singular locus of the Chow complex of a space curve and we compute its bidegree. Section 4 considers Hurwitz complexes as well as bitangent and inflectional congruences. We give a new proof for the bidegrees of these congruences using only geometric arguments. Our proof uses the concept of projective duality, which we introduce in Section 5. For expository reasons we give an introduction to the intersection theory of $\text{Gr}(1, \mathbb{P}^3)$, Chern classes and their relations to the questions studied in this paper in Section 6. This is because most modern formulations of questions of this sort are written and studied using these techniques. We consider some example questions that can be easily computed using intersection theory, for example we compute the number of lines bitangent to two general surfaces of given degree. At last, we study the singular loci of secant, bitangent and inflectional congruences in Section 7, and we describe computational aspects in Section 8.

This project was inspired by some of the problems in [31], in particular Problem 5 on Curves, Problem 4 on Surfaces and Problem 3 on Grassmannians. We give, throughout the article, full answers to these problems.

2 Preliminaries on Grassmannians

The Grassmannian $\text{Gr}(1, \mathbb{P}^3)$ of lines in \mathbb{P}^3 is a four-dimensional variety, which can be embedded into \mathbb{P}^5 via the Plücker embedding. A line that is spanned by two distinct points $(x_0 : x_1 : x_2 : x_3), (y_0 : y_1 : y_2 : y_3) \in \mathbb{P}^3$ is uniquely identified with $(p_{01} : p_{02} : p_{03} : p_{12} : p_{13} : p_{23})$, where p_{ij} is the minor formed of columns i and j of the matrix $\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix}$. The entries p_{ij} computed in this way are called *Plücker coordinates* and satisfy the Plücker relation $p_{01}p_{23} - p_{02}p_{13} + p_{03}p_{12} = 0$. All points in \mathbb{P}^5 satisfying this relation are the Plücker coordinates of some line. Analogously, one can also write lines as the intersection of two planes $\{x \in \mathbb{P}^3 \mid a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 = 0\}$ and $\{x \in \mathbb{P}^3 \mid b_0x_0 + b_1x_1 + b_2x_2 + b_3x_3 = 0\}$, and consider the minors q_{ij} of the matrix $\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix}$. These minors satisfy the same Plücker relation and are called *dual Plücker coordinates*. We can go from one convention to the other via the map

$$p_{01} \mapsto q_{23}, \quad p_{02} \mapsto -q_{13}, \quad p_{03} \mapsto q_{12}, \quad p_{12} \mapsto q_{03}, \quad p_{13} \mapsto -q_{02}, \quad p_{23} \mapsto q_{01}.$$

We study subvarieties of this Grassmannian $\text{Gr}(1, \mathbb{P}^3)$. An introduction to subvarieties of general Grassmannians is given in [1]. In particular, we are interested in their singular loci and in computing their *degree*. There is a natural notion of degree for a subvariety $\Sigma \subset \text{Gr}(1, \mathbb{P}^3)$. Formally this can be defined as the coefficients of the class of Σ in the Chow ring of $\text{Gr}(1, \mathbb{P}^3)$ with respect to the basis given by Schubert cycles. Details about this can be found in Section 6. For now, we give the following, more geometric, definitions.

Threefolds in $\text{Gr}(1, \mathbb{P}^3)$ are classically called *line complexes*. Let $H \subset \mathbb{P}^3$ be a general plane and let $v \in H$ be a general point. The degree of a line complex $\Sigma \subset \text{Gr}(1, \mathbb{P}^3)$ is defined as the number of lines $L \in \Sigma$ with $v \in L \subset H$. An example of a line complex is the Chow complex $\text{CH}_0(C)$ of a curve $C \subset \mathbb{P}^3$. A general plane H intersects C in $\deg(C)$ many points, so that there are $\deg(C)$ many lines in H that pass through a general point v in H and intersect C (see Fig. 1). Hence, the degree of the Chow complex is equal to the degree of the curve.

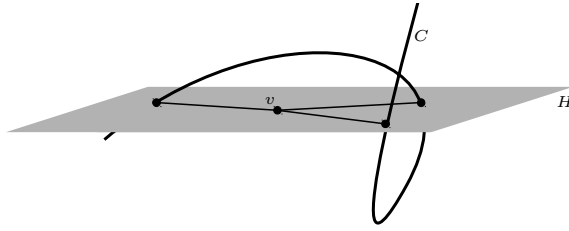


Fig. 1 $\deg \text{CH}_0(C) = \deg C$.

Surfaces in $\Sigma \subset \text{Gr}(1, \mathbb{P}^3)$ are classically known as *congruences*. The degree of a congruence is in fact a *bidegree* (α, β) . The *order* α is the number of lines $L \in \Sigma$ that pass through a general point $v \in \mathbb{P}^3$, whereas the *class* β is the number of lines $L \in \Sigma$ that lie in a general plane $H \subset \mathbb{P}^3$. Consider for example the congruence of lines passing through a fixed point x . Given a general point p , this congruence contains a unique line passing through p (the line spanned by x and p). Given a plane H , we have in general $x \notin H$, and the congruence does not contain any line contained in H . This shows that the set of lines passing through a fixed point is a congruence with bidegree $(1, 0)$. Analogously, the congruence of lines lying in a fixed plane has bidegree $(0, 1)$.

The degree of a curve $\Sigma \subset \text{Gr}(1, \mathbb{P}^3)$ is the number of lines $L \in \Sigma$ intersecting a general line of \mathbb{P}^3 . Equivalently it is the number of lines in the intersection of Σ with the Chow complex of a general line. An example of a curve in $\text{Gr}(1, \mathbb{P}^3)$ is the set of all lines that lie in a fixed plane $H \subset \mathbb{P}^3$ and pass through a fixed point $v \in H$. This curve is in fact a line in $\text{Gr}(1, \mathbb{P}^3)$. We see directly that its degree is one since a general line has one intersection point with H and there is exactly one line passing through this intersection point and v .

Finally, the degree of a zero-dimensional subvariety $\Sigma \subset \text{Gr}(1, \mathbb{P}^3)$ is simply the number of lines $L \in \Sigma$.

3 Secants

In this section we compute the singular locus of the Chow complex of a space curve, and we give a formula for the bidegree of the secant congruence in case the curve has mild singularities.

A curve $C \subset \mathbb{P}^3$ is defined by two or more homogeneous polynomials, and the polynomials used to describe the curve are not unique. There is however a way to describe C with a single polynomial equation. For this, we consider the *Chow complex*

$$\text{CH}_0(C) := \{L \mid C \cap L \neq \emptyset\} \subset \text{Gr}(1, \mathbb{P}^3).$$

This is a line complex, i.e., a hypersurface in $\text{Gr}(1, \mathbb{P}^3)$. It is defined by one polynomial in the Plücker coordinates of $\text{Gr}(1, \mathbb{P}^3)$, which is unique up to scaling and the Plücker relation. This polynomial is known as the *Chow form* of C . A nice introduction to Chow forms is given in [6].

Example 3.1 ([6, Sec. 1.2, Prop. 1.2]). The *twisted cubic* is parametrized by $(s^3 : s^2t : st^2 : t^3)$. The line L given by the intersection of the planes $\{x \in \mathbb{P}^3 \mid a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 = 0\}$ and $\{x \in \mathbb{P}^3 \mid b_0x_0 + b_1x_1 + b_2x_2 + b_3x_3 = 0\}$ intersects the twisted cubic if and only if there exists $(s : t) \in \mathbb{P}^1$ such that

$$a_0s^3 + a_1s^2t + a_2st^2 + a_3t^3 = 0 = b_0s^3 + b_1s^2t + b_2st^2 + b_3t^3. \quad (1)$$

Hence, we need to compute the *resultant* of the two cubic polynomials in (1), i.e., the polynomial in the coefficients a_i, b_j that vanishes exactly when the two polynomials have a common root. In this case, this is the determinant of the *Sylvester matrix*

$$\begin{pmatrix} a_3 & a_2 & a_1 & a_0 & 0 & 0 \\ 0 & a_3 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & a_3 & a_2 & a_1 & a_0 \\ b_3 & b_2 & b_1 & b_0 & 0 & 0 \\ 0 & b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & 0 & b_3 & b_2 & b_1 & b_0 \end{pmatrix}.$$

This determinant can be expressed in the dual Plücker coordinates q_{ij} that are the minors of the matrix $\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \end{pmatrix}$. It is equal to the determinant of the 3×3 -matrix

$$\begin{pmatrix} q_{01} & q_{02} & q_{03} \\ q_{02} & q_{03} + q_{12} & q_{13} \\ q_{03} & q_{13} & q_{23} \end{pmatrix},$$

which is the Chow form of the twisted cubic.

Whenever C has degree at least two, the set of lines that intersect the curve C at two points is a congruence, i.e., a surface in $\text{Gr}(1, \mathbb{P}^3)$. This is called the *secant congruence* of C . More precisely it is defined as

$$\text{Sec}(C) := \overline{\{L \mid \exists x, y \in \text{Reg}(C) : x, y \in L, x \neq y\}} \subset \text{Gr}(1, \mathbb{P}^3),$$

where $\text{Reg}(X)$ denotes the set of smooth points of a projective variety X . We stress that a line intersecting the curve only at a singular point might not be contained in the secant congruence, although it has intersection multiplicity at least two with the curve at that point (see Remark 3.8). For smooth curves of degree at least two the secant congruence is the singular locus of the Chow complex. For singular curves, the singular locus of the Chow complex contains one additional component for each singular point, namely all the lines passing through the point. Our key ingredient to prove this is Proposition 3.2. Note that, since in this paper we only work with varieties embedded in a projective space, by tangent space $T_x X$ of a variety $X \subset \mathbb{P}^n$ at a point $x \in X$ we will always mean the *embedded tangent space*

$$T_x X := \left\{ y \in \mathbb{P}^n \mid \forall f \in I(X) : \sum_{i=0}^n \frac{\partial f}{\partial X_i}(x) \cdot y_i = 0 \right\}.$$

Proposition 3.2. *Let X, Y be irreducible projective varieties with a birational, finite and surjective morphism $f : X \rightarrow Y$, and let $y \in Y$. Then Y is smooth at y if and only if the fiber $f^{-1}(y)$ contains exactly one point $x \in X$, X is smooth at x and the differential $d_x f : T_x X \rightarrow T_y Y$ is an injection.*

We will prove this with the help of Zariski's Main and Connectedness Theorems as well as criteria for finite maps and isomorphisms.

Theorem 3.3 (Zariski's Main Theorem). *Let Y be a normal irreducible variety and let $f : X \rightarrow Y$ be a birational morphism with finite fibers from an irreducible variety X to Y . Then f is an isomorphism of X with an open subset of Y .*

Theorem 3.4 (Zariski's Connectedness Theorem). *Let X, Y be irreducible projective varieties with a birational morphism $f : X \rightarrow Y$, and let $y \in Y$. If Y is normal at y , then $f^{-1}(y)$ is a connected set in the Zariski topology.*

Lemma 3.5 (Criterion for Finite Maps). *Let X, Y be irreducible projective varieties with a morphism $f : X \rightarrow Y$. Let $Y_0 \subset Y$ be open, $X_0 := f^{-1}(Y_0)$, and let f_0 be the restriction of f to X_0 . Then f_0 is finite if and only if its fibers are finite.*

Theorem 3.6 (Criterion for Isomorphism). *Let X, Y be irreducible varieties with a finite map $f : X \rightarrow Y$. Then f is an isomorphism if and only if it is bijective and the differential $d_x f : T_x X \rightarrow T_{f(y)} Y$ is an injection for all $x \in X$.*

The above formulation of Zariski's Main and Connectedness Theorems can for example be found in [22, Ch. III.9, pp. 288–289, Thm. I, Thm. V], whereas the two criteria appear in [14, Lem. 14.8, Thm 14.9].

Proof (Proposition 3.2). First assume that Y is smooth at y . Then it follows from Theorem 3.4 that the fiber $f^{-1}(y)$ is connected. Since f is finite, it has finite fibers and $f^{-1}(y) = \{x\}$ for some $x \in X$. Let Y_0 be the open set of smooth points in Y , and let $X_0 := f^{-1}(Y_0)$. Then the restriction of f to X_0 is an isomorphism of X_0 with Y_0 by Theorem 3.3. In particular, we have that $x \in X_0 \subset X$ is a smooth point. Moreover, Theorem 3.6 yields that the differential $d_x f$ is an injection.

For the other direction, we assume that $f^{-1}(y) = \{x\}$ for a smooth point $x \in X$ with injective differential $d_x f$. Let Y_1 be an open neighborhood of y containing only points in Y with one-element fibers and injective corresponding differentials. Then, by Theorem 3.6 and Lemma 3.5, we get an isomorphism of $X_1 := f^{-1}(Y_1)$ with Y_1 . Since $x \in X_1$ is smooth, we also have that $y \in Y_1 \subset Y$ is smooth. \square

Having this, we can describe the singular locus of the Chow complex of a given space curve.

Theorem 3.7. *Let $C \subset \mathbb{P}^3$ be an irreducible curve of degree at least two. Then we have*

$$\text{Sing}(\text{CH}_0(C)) = \text{Sec}(C) \cup \bigcup_{x \in \text{Sing}(C)} \{L \in \text{Gr}(1, \mathbb{P}^3) \mid x \in L\}.$$

Proof. We will first show that the incidence variety

$$A_C := \{(x, L) \mid x \in L\} \subset C \times \text{Gr}(1, \mathbb{P}^3)$$

is smooth at (x, L) if and only if $x \in C$ is smooth. For this, let f_1, \dots, f_k be generators for the vanishing ideal of C , and consider the affine chart where $x_0 \neq 0$ and the Plücker coordinate $p_{01} \neq 0$. Then we may assume $x = (1 : \alpha : \beta : \gamma)$ and L is spanned by $(1 : 0 : a : b)$ and $(0 : 1 : c : d)$. We have that $x \in L$ if and only if L is the row space of

$$\begin{pmatrix} 1 & \alpha & \beta & \gamma \\ 0 & 1 & c & d \end{pmatrix} = \begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \beta - \alpha c & \gamma - \alpha d \\ 0 & 1 & c & d \end{pmatrix},$$

which is equivalent to $a = \beta - \alpha c$ and $b = \gamma - \alpha d$. Hence, in the described affine chart, A_C can be written as

$$\{(\alpha, \beta, \gamma, a, b, c, d) \mid f_i(1, \alpha, \beta, \gamma) = 0 \text{ for } i = 1, \dots, k, a = \beta - \alpha c, b = \gamma - \alpha d\}. \quad (2)$$

As A_C has dimension three, it is smooth at (x, L) if and only if its tangent space, that is the kernel of the Jacobian matrix, has dimension three or, equivalently, if the Jacobian matrix has rank four. This matrix has the form

$$J_{A_C} := \begin{pmatrix} \frac{\partial f_1}{\partial \alpha}(1, \alpha, \beta, \gamma) & \frac{\partial f_1}{\partial \beta}(1, \alpha, \beta, \gamma) & \frac{\partial f_1}{\partial \gamma}(1, \alpha, \beta, \gamma) & 0 & 0 & 0 & 0 \\ & \vdots & & \vdots & \vdots & & \\ \frac{\partial f_k}{\partial \alpha}(1, \alpha, \beta, \gamma) & \frac{\partial f_k}{\partial \beta}(1, \alpha, \beta, \gamma) & \frac{\partial f_k}{\partial \gamma}(1, \alpha, \beta, \gamma) & 0 & 0 & 0 & 0 \\ & -c & 1 & 0 & -1 & 0 & -\alpha & 0 \\ & -d & 0 & 1 & 0 & -1 & 0 & -\alpha \end{pmatrix}.$$

Therefore it has rank four if and only if the Jacobian matrix of C has rank two or, in other words, if and only if $x \in C$ is smooth. This shows that A_C is smooth at (x, L) exactly when x is smooth in C .

By Lemma 3.5, the projection

$$\begin{aligned} \pi : A_C &\longrightarrow \text{CH}_0(C), \\ (x, L) &\longmapsto L \end{aligned} \quad (3)$$

is finite, since otherwise C would contain a line, contradicting our assumptions. Moreover, the general fiber of π has cardinality one as the general line $L \in \text{CH}_0(C)$ intersects C in a single point. Hence, π is birational. Applying Proposition 3.2 on π shows that $\text{CH}_0(C)$ is smooth at L if and only if $\pi^{-1}(L) = \{(x, L)\}$ where $x \in C$ is a smooth point and the differential $d_{(x, L)}\pi$ is injective. Using the same affine chart as above, we have that the differential $d_{(x, L)}\pi$ projects every element in the kernel of J_{A_C} on its last four coordinates. This map is not injective if and only if the kernel of J_{A_C} contains an element of the form $(*, *, *, 0, 0, 0, 0)^T \neq 0$. Such an element is in the kernel of J_{A_C} if and only if it is equal to $(\tau, c\tau, d\tau, 0, 0, 0, 0)^T$ for $\tau \in \mathbb{C} \setminus \{0\}$ and

$$\left(\frac{\partial f_i}{\partial \alpha} + c \frac{\partial f_i}{\partial \beta} + d \frac{\partial f_i}{\partial \gamma} \right) (1, \alpha, \beta, \gamma) = 0 \text{ for } i = 1, \dots, k.$$

Hence, for a smooth point $x \in C$, the differential $d_{(x, L)}\pi$ is not injective if and only if L is the tangent line of C at x .

Since we have that $|\pi^{-1}(L)| = 1$ if and only if L is not a secant line and since all tangent lines to C are contained in $\text{Sec}(C)$, it follows that $\text{CH}_0(C)$ is smooth at L if and only if $L \notin \text{Sec}(C)$ and L meets C at a smooth point. \square

Remark 3.8. We can see with local computations that the secant congruence of C generally does not contain all lines through singular points of C . For this, let $x \in C$ be an *ordinary singularity*, i.e., x is the intersection of $r \geq 2$ branches of C and the r tangent lines of the branches are pairwise different. We will now show that a line L intersecting C only at x is contained in $\text{Sec}(C)$ if and only if L lies in a plane spanned by two of the r tangent lines at x . The union of all those lines form the so called *tangent star* of C at x . This notion has been introduced in [18] and studied in [28].

Assume $x = (1 : 0 : 0 : 0)$ and $L \in \text{Sec}(C)$ intersects C only at x . The line L must be the limit of lines L_t that intersect C at two distinct smooth points. Without loss of generality, L is not one of the tangent lines of C at x and each line L_t intersects C at at least two distinct branches. Since there are only finitely many branches, we can further assume that the lines L_t intersect the same two branches of C . These two branches are parametrized by $(1 : f_1(t) : f_2(t) : f_3(t))$ and $(1 : g_1(t) : g_2(t) : g_3(t))$, respectively, with $f_i(0) = 0 = g_j(0)$. Then the two tangent lines at these branches are spanned by x and $(1 : f'_1(0) : f'_2(0) : f'_3(0))$ or $(1 : g'_1(0) : g'_2(0) : g'_3(0))$, respectively. The lines L_t intersect the first branch at $(1 : f_1(\varphi(t)) : f_2(\varphi(t)) : f_3(\varphi(t)))$ and the second branch at $(1 : g_1(\psi(t)) : g_2(\psi(t)) : g_3(\psi(t)))$, where $\varphi(0) = 0 = \psi(0)$. The Plücker coordinates of L_t are

$$\left(\frac{g_1(\psi(t)) - f_1(\varphi(t))}{t} : \frac{g_2(\psi(t)) - f_2(\varphi(t))}{t} : \frac{g_3(\psi(t)) - f_3(\varphi(t))}{t} : \frac{f_1(\varphi(t))g_2(\psi(t)) - f_2(\varphi(t))g_1(\psi(t))}{t} : \dots \right).$$

Letting t go to zero, we get the line L with Plücker coordinates

$$(g'_1(0)\psi'(0) - f'_1(0)\varphi'(0) : g'_2(0)\psi'(0) - f'_2(0)\varphi'(0) : g'_3(0)\psi'(0) - f'_3(0)\varphi'(0) : 0 : 0 : 0).$$

This line is spanned by x and

$$(1 : g'_1(0)\psi'(0) - f'_1(0)\varphi'(0) : g'_2(0)\psi'(0) - f'_2(0)\varphi'(0) : g'_3(0)\psi'(0) - f'_3(0)\varphi'(0)),$$

and thus lies in the plane spanned by the two tangent lines.

Moreover, we see from this computation that all lines that pass through x and lie in the plane spanned by the tangent lines can be approximated by lines that intersect both of the branches at points different from x . For this, one only needs to choose $\varphi(t) = \lambda t$ and $\psi(t) = \mu t$ for all possible $\lambda, \mu \in \mathbb{C} \setminus \{0\}$.

The bidegree of the secant congruence of a smooth curve was calculated in [2, Prop. 2.1] with Chern classes. Here we give a geometric description of this bidegree also for curves with ordinary singularities. As in Remark 3.8, an ordinary singularity is the intersection of $r \geq 2$ branches of the curve and the r tangent lines of the branches are pairwise different. The number r is called the multiplicity of the ordinary singularity. If $r = 2$, the singularity is called a (*simple*) *node*.

Theorem 3.9. *Let $C \subset \mathbb{P}^3$ be an irreducible curve that is not contained in any plane and has only ordinary singularities x_1, \dots, x_s with multiplicities r_1, \dots, r_s . Let d denote the degree and g denote the geometric genus of the curve. Then the bidegree of the secant congruence $\text{Sec}(C)$ is $(\frac{1}{2}(d-1)(d-2) - g - \sum_{i=1}^s \frac{1}{2}r_i(r_i-1), \frac{1}{2}d(d-1))$.*

Proof. Let $H \subset \mathbb{P}^3$ be a general plane. The intersection of H with C consists of d points. Any two of these points define a secant line lying in H (see Fig. 2). Hence, there are $\binom{d}{2}$ secant lines contained in H . This is the class of $\text{Sec}(C)$.

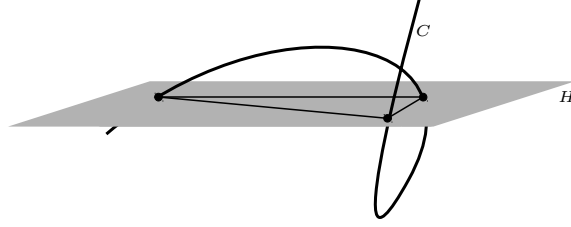


Fig. 2 $\text{class}(\text{Sec}(C)) = \frac{1}{2} \deg(C) (\deg(C) - 1)$.

Let $v \in \mathbb{P}^3$ be a general point. We are interested in calculating the order of $\text{Sec}(C)$, which is the number of secant lines passing through v . The point v defines a projection $\pi_v : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$. Let C' be the image of C under π_v . A line passing through v and intersecting C at two points gets mapped by π_v to a simple node of the plane curve C' (see Fig. 5). Every ordinary singularity of C gets mapped to an ordinary singularity of C' with the same multiplicity. The plane curve C' has the same degree as the space curve C . Moreover, as the geometric genus is invariant under birational transformation (see [15, Thm. II.8.19]), it also has the same genus. By the genus-degree formula for plane curves [27, p. 54, Eq. (7)], the genus g is equal to $\frac{1}{2}(d-1)(d-2) - \sum_{i=1}^s \frac{1}{2}r_i(r_i-1)$ minus the number of secants of C passing through v . This concludes the proof. \square

Remark 3.10. If $C \subset \mathbb{P}^3$ is a curve of degree $d \geq 2$ which is contained in a plane, then its secant congruence consists of all lines in that plane and thus has bidegree $(0, 1)$.

Problem 5 on Curves in [31] asks to compute the dimension and bidegree of $\text{Sing}(\text{CH}_0(C))$. We already know by Theorem 3.7 that its dimension is two if C is not a line. For completeness we also state its bidegree explicitly.

Corollary 3.11. *Let $C \subset \mathbb{P}^3$ be an irreducible curve of degree $d \geq 2$ that has only ordinary singularities x_1, \dots, x_s with multiplicities r_1, \dots, r_s . Let g denote the geometric genus of the curve. Then the bidegree of $\text{Sing}(\text{CH}_0(C))$ is equal to*

$$\left(\frac{1}{2}(d-1)(d-2) - g - \sum_{i=1}^s \frac{1}{2}r_i(r_i-1) + s, \frac{1}{2}d(d-1) \right)$$

if C is not contained in a plane, and $(s, 1)$ if C is contained in a plane.

Proof. This follows immediately from Theorems 3.7 and 3.9, Remark 3.10, and the observation that the bidegree of each congruence $\{L \in \text{Gr}(1, \mathbb{P}^3) \mid x_i \in L\}$ is $(1, 0)$. \square

4 Bitangents and Inflections

By analogy with lines meeting a space curve, we consider in what follows the set of lines tangent to a surface $S \subset \mathbb{P}^3$. If the surface is not a plane, this defines again a hypersurface/line complex in $\text{Gr}(1, \mathbb{P}^3)$:

$$\text{CH}_1(S) := \overline{\{L \mid \exists x \in \text{Reg}(S) : x \in L \subset T_x S\}} \subset \text{Gr}(1, \mathbb{P}^3).$$

The defining equation for this hypersurface is called the *Hurwitz form* of S [30]. A generalization of Chow and Hurwitz forms to arbitrary projective varieties $X \subset \mathbb{P}^n$ is studied in [11, Ch. 3.3+4.3] and [19], and it is called the *i-th coisotropic variety* of X , where $0 \leq i \leq \dim X$:

$$\text{CH}_i(X) := \overline{\{L \mid \exists x \in \text{Reg}(X) : x \in L, \dim(L \cap T_x X) \geq i\}} \subset \text{Gr}(n - \dim X + i - 1, \mathbb{P}^n).$$

Equivalently, the coisotropic varieties of X are defined to be the closure of the set of all linear subspaces of a fixed dimension that have a non-transversal intersection with the given variety X at a smooth point. It is shown in [19] that $\text{CH}_i(X)$ is a hypersurface in the Grassmannian $\text{Gr}(n - \dim X + i - 1, \mathbb{P}^n)$ if and only if $i \leq \dim X - \text{codim} X^\vee + 1$, where X^\vee is the *projectively dual variety* to X (see Section 5 for the definition of X^\vee). A very nice example of a Hurwitz form is related to Problem 3 on Curves in [31]: the Hurwitz form of the Veronese surface $V \subset \mathbb{P}^5$ (i.e., the defining polynomial of the hypersurface $\text{CH}_1(V) \subset \text{Gr}(3, \mathbb{P}^5)$) is the *tact invariant*, which is the polynomial in the coefficients of two plane conics that vanishes exactly when the conics are tangent [30, Ex. 2.7].

Example 4.1. Consider the Veronese embedding

$$\begin{aligned} \phi : \mathbb{P}^2 &\longrightarrow \mathbb{P}^5, \\ (x : y : z) &\longmapsto (x^2 : y^2 : z^2 : yz : xz : xy). \end{aligned}$$

Two conics $\mathcal{C}_1, \mathcal{C}_2 \subset \mathbb{P}^2$ are tangent to each other if and only if their images $\phi(\mathcal{C}_1), \phi(\mathcal{C}_2)$ are tangent. Note that the image $\phi(\mathcal{C}_i)$ is simply the intersection of the Veronese surface $V := \phi(\mathbb{P}^2)$ with a hyperplane $H_i \subset \mathbb{P}^5$. Hence, the conics $\mathcal{C}_1, \mathcal{C}_2$ are tangent if and only if there is point $x \in V \cap H_1 \cap H_2$ at which the two tangent lines $T_x V \cap H_1$ and $T_x V \cap H_2$ coincide. In terms of the three dimensional subspace $L := H_1 \cap H_2$ this is equivalent to $x \in V \cap L$ and $\dim(T_x V \cap L) \geq 1$. This shows that the tact invariant is the Hurwitz form of V .

Analogously to the case of secants, the set of lines tangent to a surface $S \subset \mathbb{P}^3$ at two smooth points forms a congruence

$$\text{Bit}(S) := \overline{\{L \mid \exists x, y \in \text{Reg}(S) : x, y \in L \subset T_x S \cap T_y S, x \neq y\}} \subset \text{Gr}(1, \mathbb{P}^3),$$

which we call the *bitangent congruence* of S . To a surface S we also associate its *inflectional locus*

$$\text{Infl}(S) := \overline{\{L \mid \exists x \in \text{Reg}(S) : L \text{ intersects } S \text{ at } x \text{ with multiplicity } 3\}} \subset \text{Gr}(1, \mathbb{P}^3).$$

For general S this is a congruence, and we will refer to it as *inflectional congruence* when it has dimension two. This is not always the case though, as we see for example in Remark 5.6. For a smooth surface S of degree at least four that does not contain any lines, we show in the following that the above two congruences form the singular locus of the Hurwitz complex of S . Before we state the theorem we stress that, if the degree d of S is three or less and a line L is bitangent to S , then L is contained in S . This is a consequence of Bézout's Theorem: if L is not contained in S , then $L \cap S$ is made of $d < 4$ points, counted with multiplicity. On the other hand, when the degree of S is four or more, the assumption that S does not contain any line is very mild. It is a classical result that this holds for a general surface of degree $d \geq 4$ in \mathbb{P}^3 (see e.g. [33]). By general surface of degree d in \mathbb{P}^3 we mean, as is custom, a surface defined by a polynomial corresponding to a general point in $\mathbb{P}(\mathbb{C}[x_0, x_1, x_2, x_3]_d)$.

Theorem 4.2. *Let $S \subset \mathbb{P}^3$ be an irreducible smooth surface of degree at least four which does not contain any lines. Then we have $\text{Sing}(\text{CH}_1(S)) = \text{Bit}(S) \cup \text{Infl}(S)$.*

Proof. We will first show that the incidence variety

$$A_S := \{(x, L) \mid x \in L \subset T_x S\} \subset S \times \text{Gr}(1, \mathbb{P}^3)$$

is smooth. To see this, we denote the defining equation for S by f and consider the affine chart where $x_0 \neq 0$ and the Plücker coordinate $p_{01} \neq 0$. Then we may assume $x = (1 : \alpha : \beta : \gamma)$ and L is spanned by $(1 : 0 : a : b)$ and $(0 : 1 : c : d)$. In this affine chart, S is defined by $g_0(\alpha, \beta, \gamma) := f(1, \alpha, \beta, \gamma)$. As in the proof of Theorem 3.7, we have that $x \in L$ if and only if $a = \beta - \alpha c$ and $b = \gamma - \alpha d$. For such an (x, L) we further have that $L \subset T_x S$ if and only if $(0 : 1 : c : d) \in T_x S$. This is characterized by $g_1 := \frac{\partial g_0}{\partial \alpha} + c \frac{\partial g_0}{\partial \beta} + d \frac{\partial g_0}{\partial \gamma} = 0$. Hence, in the described affine chart, A_S can be written as

$$\{(\alpha, \beta, \gamma, a, b, c, d) \mid g_0 = 0, a = \beta - \alpha c, b = \gamma - \alpha d, g_1 = 0\} \quad (4)$$

and its Jacobian matrix has the form

$$J_{A_S} := \begin{pmatrix} \frac{\partial g_0}{\partial \alpha} & \frac{\partial g_0}{\partial \beta} & \frac{\partial g_0}{\partial \gamma} & 0 & 0 & 0 & 0 \\ -c & 1 & 0 & -1 & 0 & -\alpha & 0 \\ -d & 0 & 1 & 0 & -1 & 0 & -\alpha \\ \frac{\partial g_1}{\partial \alpha} & \frac{\partial g_1}{\partial \beta} & \frac{\partial g_1}{\partial \gamma} & 0 & 0 & \frac{\partial g_0}{\partial \beta} & \frac{\partial g_0}{\partial \gamma} \end{pmatrix}.$$

The matrix J_{A_S} has rank four everywhere on (4), since S is smooth. This shows that A_S is smooth as well.

Now we consider the projection

$$\begin{aligned}\pi : A_S &\longrightarrow \text{CH}_1(S), \\ (x, L) &\longmapsto L.\end{aligned}$$

Since S does not contain any lines, all fibers of π are finite sets and π is finite by Lemma 3.5. Moreover the general fiber of π has cardinality one, so π is birational. It follows from Proposition 3.2 that a tangent line L is smooth in $\text{CH}_1(S)$ if and only if the fiber $\pi^{-1}(L)$ consists of one point (x, L) and the differential $d_{(x,L)}\pi$ is injective. We have that $|\pi^{-1}(L)| = 1$ if and only if L is not a bitangent.

It is left to show that the differential $d_{(x,L)}\pi$ is injective if and only if L is a simple tangent of S at x . For this, we use the same affine chart as above and the description (4) for A_S . Then we have that the differential $d_{(x,L)}\pi$ projects every element in the kernel of J_{A_S} on its last four coordinates. This map is not injective if and only if the kernel of J_{A_S} contains an element of the form $(*, *, *, 0, 0, 0, 0)^T \neq 0$. Such an element is in the kernel of J_{A_S} if and only if it is equal to $(\tau, c\tau, d\tau, 0, 0, 0, 0)^T$ for $\tau \in \mathbb{C} \setminus \{0\}$ and

$$g_1(\alpha, \beta, \gamma) = g_2(\alpha, \beta, \gamma) = 0, \quad (5)$$

where $g_2 := \frac{\partial g_1}{\partial \alpha} + c \frac{\partial g_1}{\partial \beta} + d \frac{\partial g_1}{\partial \gamma}$. Parametrizing the line L by $\ell(u, v) := (u : u\alpha + v : u\beta + v\gamma : u\gamma + v\delta)$ for $(u : v) \in \mathbb{P}^1$ shows that L intersects S with multiplicity at least three at x if and only if $f(\ell(u, v))$ is divisible by v^3 . This is equivalent to the condition that $\frac{\partial f(\ell(u, v))}{\partial v} = \frac{\partial^2 f(\ell(u, v))}{\partial^2 v} = 0$ whenever $v = 0$, which means exactly that (5) is satisfied. \square

The bidegrees of the bitangent and the inflectional congruence of a general surface are calculated in [2, Prop. 3.3] with Chern classes. Using a similar approach, the bidegree of the inflectional congruence was already computed in [23, Prop. 4.1]. We will give a purely geometric proof for these bidegrees. Our geometric ideas use the concept of *projective duality*. Section 5 is devoted to this important notion and contains our proof for the following theorem, which solves Problem 4 on Surfaces in [31].

Theorem 4.3. *Let $S \subset \mathbb{P}^3$ be a general smooth irreducible surface of degree $d \geq 4$. Then the bidegree of the bitangent congruence $\text{Bit}(S)$ is $(\frac{1}{2}d(d-1)(d-2)(d-3), \frac{1}{2}d(d-2)(d-3)(d+3))$, and the bidegree of the inflectional congruence $\text{Infl}(S)$ is $(d(d-1)(d-2), 3d(d-2))$.*

5 Duality

Every projective variety X has an associated *dual variety* X^\vee . As we will see in the following, this duality satisfies a lot of desirable properties. A classical reference for

this geometrical notion is [11]. The definition of duality generalizes the orthogonal complement of linear subspaces.

By $(\mathbb{P}^n)^*$ we denote the projectivization of the dual vector space $(\mathbb{C}^{n+1})^*$, which is formed by hyperplanes in \mathbb{P}^n . The two projective spaces \mathbb{P}^n and $(\mathbb{P}^n)^*$ can be identified by sending every point $y = (y_0 : \dots : y_n) \in \mathbb{P}^n$ to the hyperplane

$$\{y = 0\} := \left\{ x \in \mathbb{P}^n \mid \sum_{i=0}^n y_i x_i = 0 \right\} \in (\mathbb{P}^n)^*.$$

A hyperplane in \mathbb{P}^n is called *tangent* to X at a smooth point $x \in X$ if it contains the embedded tangent space $T_x X$. The closure X^\vee of the set of all hyperplanes that are tangent to X at some smooth $x \in X$ is called the *projectively dual variety* of X :

$$X^\vee := \overline{\{H \mid \exists x \in \text{Reg}(X) : T_x X \subset H\}} \subset (\mathbb{P}^n)^*.$$

In the case that $X = \mathbb{P}(V)$ is the projectivization of a linear subspace $V \subset \mathbb{C}^{n+1}$, the dual $(\mathbb{P}(V))^\vee$ is the set of all hyperplanes containing $\mathbb{P}(V)$, which is exactly the projectivization of the orthogonal complement $V^\perp \subset (\mathbb{C}^{n+1})^*$ (not with respect to the complex scalar product $(x, y) \mapsto \sum x_i \bar{y}_i$, but with respect to the non-degenerate bilinear form $(x, y) \mapsto \sum x_i y_i$). Note that the dual $(\mathbb{P}(V))^\vee$ is *not* the projectivization of the dual vector space V^* . In particular, $(\mathbb{P}^n)^\vee$ is the empty set.

It is known for all irreducible varieties X that X^\vee is also an irreducible variety [11, Ch. 1, Prop. 1.3]. We will use the identification of \mathbb{P}^n and its dual space $(\mathbb{P}^n)^*$ to view the dual variety X^\vee as a subvariety of \mathbb{P}^n . With this, we will also make frequent use of the following biduality of projective varieties over \mathbb{C} , which is also known as reflexivity.

Theorem 5.1 ([11, Ch. 1, Thm. 1.1]). *For every irreducible variety $X \subset \mathbb{P}^n$, we have $(X^\vee)^\vee = X$. More precisely, if $x \in X$ is smooth and $H \in X^\vee$ is smooth, then H is tangent to X at x if and only if x – regarded as a hyperplane in $(\mathbb{P}^n)^*$ – is tangent to X^\vee at H .*

The projectively dual of a plane curve of degree at least two is again a plane curve, and the dual of a line in the plane is a point. The projectively dual of a space curve of degree at least two is a surface, whereas the dual of a space line is a line. The dual of a surface of degree at least two in \mathbb{P}^3 can be either a curve or a surface, whereas the dual of a plane in \mathbb{P}^3 is a point.

We have two main results in this section. First, we will prove Theorem 4.3. Secondly, we will relate the congruences associated to a curve/surface in \mathbb{P}^3 with the congruences associated to its projectively dual variety.

5.1 Proof of Theorem 4.3

We note that many of the following proof ideas can be found in [24, p. 230]. The key tool for our proof for the bidegrees of congruences associated to a surface is

Plücker's formula (see [7, Thm. 1.2.7, Eq. (1.50)]). It computes the degree of the dual curve of a given plane curve with mild singularities. Such a mild singularity can be either a node or a *cusp*. Intuitively a cusp is a point where a moving point on the curve has to change its direction (see Fig. 4 and, for a formal definition, [13, p. 277]).

Proposition 5.2 (Plücker's formula). *Let $C \subset \mathbb{P}^2$ be an irreducible curve of degree d which has exactly κ cusps, ν simple nodes and no other singularities. Then the degree of the dual curve C^\vee can be computed as follows:*

$$\deg(C^\vee) = d(d-1) - 3\kappa - 2\nu.$$

Proof (Sketch). Let C be defined by a polynomial f of degree d . First we assume that C is smooth, i.e., $\kappa = \nu = 0$. The degree of the dual plane curve $C^\vee \subset (\mathbb{P}^2)^*$ is the number of points of C^\vee lying on a general line $L \subset (\mathbb{P}^2)^*$. Equivalently, this degree is the number of tangent lines to C passing through a general point $y \in \mathbb{P}^2$. Such a tangent line at $x \in C$ passes through y if and only if $g(x) := \sum_{i=0}^2 \frac{\partial f}{\partial x_i}(x) \cdot y_i = 0$. Hence, the degree of C^\vee is the number of points $x \in V(f, g)$ – the vanishing set of f and g . This finite set contains $d(d-1)$ many points since $\deg(g) = d-1$.

If C is singular, the degree of C^\vee is the number of lines that are tangent to C at a *smooth* point and that pass through the general point y . Those smooth points are contained in the set $V(f, g)$, but also all singular points lie in $V(f, g)$. Thus the inequality $\deg(C^\vee) \leq d(d-1)$ holds. The curve $V(g)$ passes through each node of C with intersection multiplicity two and through each cusp of C with intersection multiplicity three. Hence, for every node the number $d(d-1)$ overcounts by two, whereas it overcounts by three for every cusp. \diamond

Proposition 5.2 allows us to give an easy proof for the degree of the Hurwitz complex of a smooth surface. This result is also implied by [30, Thm. 1.1].

Corollary 5.3. *For an irreducible smooth surface $S \subset \mathbb{P}^3$ of degree $d \geq 2$, the degree of $\text{CH}_1(S)$ is equal to $d(d-1)$.*

Proof. Let $H \subset \mathbb{P}^3$ be a general plane and $v \in H$ be a general point. The degree of $\text{CH}_1(S)$ is the number tangent lines L to S satisfying $p \in L \subset H$. The intersection $S \cap H$ is a smooth plane curve of degree d (this is a special case of Bertini's Theorem, see e.g. [14, Thm. 17.16]). The degree of $\text{CH}_1(S)$ is the number of tangent lines to $S \cap H$ passing through the general point v (see Fig. 3). This is by definition equal to the degree of the dual plane curve $(S \cap H)^\vee$, which is $d(d-1)$ by Proposition 5.2. \square

Moreover, using Proposition 5.2, we can show the following classical result on the number of bitangents and inflectional tangents to a general smooth plane curve (in particular, a general plane quartic has 28 bitangents), and we can finally prove Theorem 4.3.

Proposition 5.4. *A general smooth irreducible plane curve of degree d has $\frac{1}{2}d(d-2)(d-3)(d+3)$ bitangents and $3d(d-2)$ inflectional tangents.*

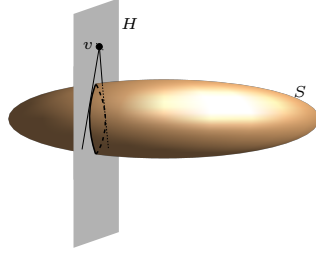


Fig. 3 $\deg(\mathrm{CH}_1(S)) = \deg(S)(\deg(S) - 1)$.

Proof. Consider a general smooth curve $C \subset \mathbb{P}^2$ of degree d . It follows from Theorem 5.1 that a bitangent to C is a singular point in C^\vee . In fact, bitangents to C correspond to nodes of C^\vee . Moreover, inflectional tangents to C correspond to cusps of C^\vee (see Fig. 4 and [13, pp. 277–278]). We know from Proposition 5.2 that C^\vee has degree $d(d-1)$. We denote by κ and v the number of cusps and nodes of C^\vee , respectively. Applying Proposition 5.2 to the plane curve C^\vee yields

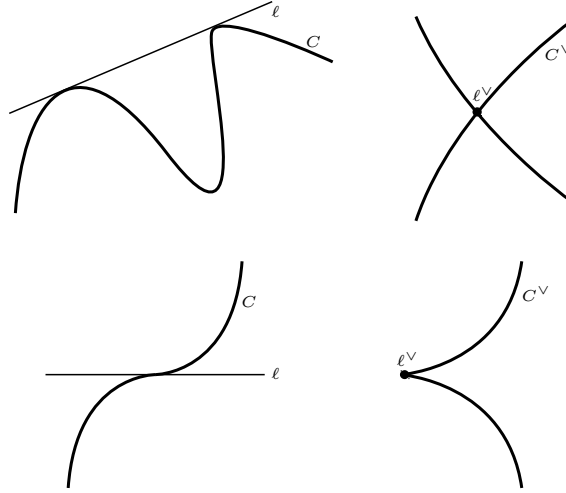


Fig. 4 A bitangent and an inflectional line corresponding to a node and a cusp of the dual curve.

$$d = d(d-1)(d(d-1)-1) - 3\kappa - 2v. \quad (6)$$

The geometric genus g of the dual curves C and C^\vee is the same [34, p. 4] and can be computed by the genus-degree formula [27, p. 54, Eq. (7)] as

$$\frac{1}{2}(d-1)(d-2) = g = \frac{1}{2}(d(d-1)-1)(d(d-1)-2) - \kappa - v. \quad (7)$$

Using (6) and (7) leads to

$$v = 3(\kappa + v) - (3\kappa + 2v) = \frac{1}{2}d(d-2)(d-3)(d+3).$$

Finally, solving for κ gives us $\kappa = 3d(d-2)$. \square

Proof (Theorem 4.3). For a general plane $H \subset \mathbb{P}^3$, the intersection $H \cap S$ is a general smooth plane curve of degree d . By Proposition 5.4, the number of bitangents to S that are contained in H is equal to $\frac{1}{2}d(d-2)(d-3)(d+3)$. This is the class of $\text{Bit}(S)$. Analogously, the class of $\text{Infl}(S)$ is the number of inflectional lines to $H \cap S$, which is $3d(d-2)$ by Proposition 5.4.

Let $y \in \mathbb{P}^3$ be a general point. It is left to calculate the number of bitangents and inflectional lines to S that pass through y . We denote the defining polynomial of S by f . Let us consider the *polar curve* $C \subset S$ with respect to the point y , which is the set of all points $x \in S$ such that the line through x and y is tangent to S at x (see Fig. 6). The condition $x \in C$ is equivalent to saying that $y \in T_x S$. As in the proof idea for Proposition 5.2, the polar curve C is the vanishing set $V(f, g)$, where $g(x) := \sum_{i=0}^3 \frac{\partial f}{\partial X_i}(x) \cdot y_i$. Thus the curve C has degree $d(d-1)$.

Now we consider the projection $\pi_y : \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ from the point y . This map restricted to S is generally $d : 1$ and is ramified over the polar curve C (i.e., the set of points where $\pi_y|_S$ is not a $d : 1$ map is exactly C). We denote by C' the image of C under π_y . A bitangent line to the surface S that passes through y contains two points of the polar curve C , and these two points are mapped to a simple node of C' by π_y (see Fig. 5). Note that all of these nodes of C' have indeed two distinct tangent lines since no bitangent line passing through y is contained in a bitangent plane which is tangent at the same two points as the line. The latter holds because the bitangent planes to S are a one-dimensional family of planes, which implies that the union of their contained bitangent lines is a surface in \mathbb{P}^3 which does not contain y due to the generality of y .

The inflectional lines to S going through y are exactly the tangent lines of C passing through y . This can be seen as follows: for $x \in S$, the line L between x and y can be parametrically represented as the set of points

$$\ell(u, v) = (ux_0 + vy_0 : ux_1 + vy_1 : ux_2 + vy_2 : ux_3 + vy_3),$$

where $(u : v) \in \mathbb{P}^1$. Then L is an inflectional tangent if and only if $f(\ell(u, v))$ is divisible by v^3 . This is equivalent to that $\frac{\partial f(\ell(u, v))}{\partial v} = \frac{\partial^2 f(\ell(u, v))}{\partial^2 v} = 0$ whenever $v = 0$, which means exactly that $x \in C$ and $\sum_{i=0}^3 \frac{\partial g}{\partial X_i}(x) \cdot y_i = 0$. The latter holds if and only if $y \in T_x C$. This shows that the inflectional lines to S passing through y are the tangents to C passing through y , and are thus mapped to the cusps of C' by π_y (see Fig. 5).

Hence, bitangent and inflectional lines to S passing through y correspond to nodes and cusps of C' , and we need to calculate the numbers v of nodes and κ of cusps of the plane curve C' . Due to generality of the surface S and the point y we can apply Proposition 5.2, which yields

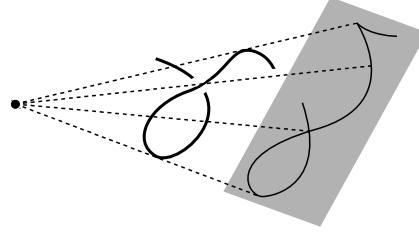


Fig. 5 Projection of a space curve from a point, showing a secant mapped to a node and a tangent mapped to a cusp.

$$\deg((C')^\vee) = \deg(C')(\deg(C') - 1) - 3\kappa - 2\nu. \quad (8)$$

The degree of C' is equal to the degree of the polar curve C , which is $d(d-1)$. To conclude the proof we first compute κ and then the degree of the projectively dual curve to C' .

Claim. $\kappa = d(d-1)(d-2)$.

Proof. Let $x \in S$. Using the same parametrization $\ell(u, v)$ of the line L between x and y as above, we have that L is an inflectional tangent to S if and only if $x \in V(f, g, h)$, where $h(x) := \sum_{i=0}^3 \frac{\partial g}{\partial x_i}(x) \cdot y_i$. This set consists of $d(d-1)(d-2)$ points for a general surface S , since the equation h has degree $d-2$. \diamond

Claim. $\deg((C')^\vee) = \deg(S^\vee)$.

Proof. By definition, the degree δ of $(C')^\vee$ is the number of tangent lines to $C' \subset \mathbb{P}^2$ going through a general point $z \in \mathbb{P}^2$. The preimage of z under the projection π_y is a line $L \subset \mathbb{P}^3$ containing y (see Fig. 6). Hence, δ is the number of tangent lines to C intersecting L (in a point different from y). Since y is the defining point for the polar curve C , we have for every line T which is tangent to C (at a point x) and intersects L that T and L span the tangent plane of S at x . On the other hand, given any plane H which is tangent to S at a point x and contains L , we have that x must lie on the polar curve C and that H is spanned by L and the tangent line T to C at x . This implies that T and L intersect. In conclusion, δ is the number of tangent planes to S containing L , which is the degree of the dual surface S^\vee . \diamond

Claim. $\deg(S^\vee) = d(d-1)^2$.

Proof. By definition, the degree of S^\vee is the number of tangent planes to S containing a general line. Alternatively, this is the number of tangent planes to S containing two general points $y, z \in \mathbb{P}^3$. Therefore this is the number of intersection points of the two polar curves of S defined by y and z , which is the set $V(f, g, \tilde{g})$, where $\tilde{g} := \sum_{i=0}^3 \frac{\partial f}{\partial x_i} z_i$. This finite set of points has degree $d(d-1)^2$. \diamond

Plugging the results of the above three claims as well as $\deg(C') = d(d-1)$ into (8) yields $\nu = \frac{1}{2}d(d-1)(d-2)(d-3)$, which concludes the proof. \square

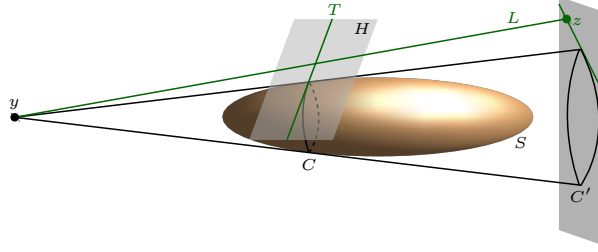


Fig. 6 Visualization for the proof of $\deg((C')^\vee) = \deg(S^\vee)$.

5.2 Congruences of Projectively Dual Varieties

We prove in the following that the secant locus of a nice enough curve is essentially the same as the bitangent congruence of its dual surface. With “essentially the same” we mean the following: for every subvariety $\Sigma \subset \text{Gr}(1, \mathbb{P}^3)$ we can define another subvariety $\Sigma^\perp \subset \text{Gr}(1, \mathbb{P}^3)$ by sending every line $L \in \Sigma$ to its dual (orthogonal) line $L^\vee \in \Sigma^\perp$. If we write a line $L \in \text{Gr}(1, \mathbb{P}^3)$ as the kernel of a 2×4 -matrix A (i.e., as the intersection of two planes) or as the row space of a 2×4 -matrix B (i.e., as the span of two points), then L^\vee will simply be the row space of A and the kernel of B . Hence, the variety Σ^\perp is merely another representation of Σ by changing from one convention to the other. It follows immediately from Theorem 5.1 that for every congruence $\Sigma \subset \text{Gr}(1, \mathbb{P}^3)$ with bidegree (α, β) , the bidegree of Σ^\perp is (β, α) .

Theorem 5.5. *Let $C \subset \mathbb{P}^3$ be an irreducible smooth curve that is not contained in any plane. Then we have $\text{Sec}(C)^\perp = \text{Bit}(C^\vee)$ and the inflectional lines of C^\vee are dual to the tangent lines of C ; in particular, $\text{Infl}(C^\vee) \subset \text{Bit}(C^\vee)$.*

Proof. First we show that $\text{Sec}(C)^\perp \subset \text{Bit}(C^\vee)$. Let L be a line that intersects C at two distinct points x, y and is neither the tangent line $T_x C$ nor $T_y C$. Then L spans with $T_x C$ a plane corresponding to a point $a \in C^\vee$. Analogously, the lines L and $T_y C$ give a point $b \in C^\vee$. We can assume that both these points are smooth in C^\vee . By Theorem 5.1, the points $a, b \in C^\vee$ must be distinct points with tangent planes corresponding to x and y , respectively. It follows that L^\vee is tangent to C^\vee at a and b .

Now we show the other direction $\text{Bit}(C^\vee) \subset \text{Sec}(C)^\perp$. For this, let L be a line that is tangent to C^\vee at two distinct smooth points $a, b \in C^\vee$. The tangent planes at a, b correspond to two points $x, y \in C$. If $x \neq y$, then L^\vee is the secant to C through these two points. If $x = y$, then L^\vee is the tangent line of C at x by Theorem 5.1.

Finally, let L be an inflectional line at a smooth point $a \in C^\vee$. We pick any point $y \in L^\vee \setminus C$. This point y corresponds to a plane H such that $L = T_a C^\vee \cap H$. Then L is still an inflectional line at the plane curve $C^\vee \cap H \subset H$. The projectively dual to L in the projective plane H is a point, which is in fact a cusp of the plane curve $(C^\vee \cap H)^\vee \subset H^*$ (cf. Fig. 4).

Claim. $(C^\vee \cap H)^\vee$ is the image $\text{pr}_y C$ of the projection of C from y (a more general statement holds also for subvarieties of \mathbb{P}^n , see [16, Prop. 6.1]).

Proof. Consider a smooth point $z \in \text{pr}_y C$. It is the projection of a smooth point of C whose tangent line does not contain y . Hence, this tangent line spans together with y a plane such that the dual point $z_{(d)}$ to this plane is contained in the plane curve $C^\vee \cap H$. Thus, the tangent line $T_z(\text{pr}_y C)$ is equal to $\text{pr}_y(z_{(d)}^\vee)$. The latter is the line in H^* which is dual to the point $z_{(d)} \in H$. (this can be easily seen from a local computation: for $y = (1 : 0 : 0 : 0)$ and $z_{(d)} = (0 : z_1 : z_2 : z_3)$, the dual line to $z_{(d)} \in H$ is defined by the equation $z_1 x_1 + z_2 x_2 + z_3 x_3 = 0$, which is the line $\text{pr}_y(z_{(d)}^\vee)$). This shows $(\text{pr}_y C)^\vee \subset C^\vee \cap H$, but since both curves are irreducible this inclusion must be an equality. \diamond

Hence, when considering L in the projective plane H , its dual point is a cusp of $\text{pr}_y C$. This implies that L^\vee must be the tangent line $T_x C$, where $x \in C$ is the point corresponding to the tangent plane $T_a C^\vee$ (see Fig. 5). All of these arguments can be reversed, showing that the dual of a tangent line to C is an inflectional line to C^\vee . Moreover, since every tangent line to C is contained in $\text{Sec}(C)$, we have the inclusion $\text{Infl}(C^\vee) \subset \text{Bit}(C^\vee)$. \square

Remark 5.6. The inflectional locus of a surface in \mathbb{P}^3 is not always a congruence. Indeed, with the assumptions of Theorem 5.5 we have that $\text{Infl}(C^\vee)$ is a curve, as $\text{Infl}(C^\vee)^\perp$ is the set of tangent lines to C .

Similarly, one can show for a curve $C \subset \mathbb{P}^3$ with dual surface $C^\vee \subset \mathbb{P}^3$ that $\text{CH}_0(C)^\perp = \text{CH}_1(C^\vee)$ [19, Thm. 20, also Ex. 21]. This equality and Theorem 5.5 show that the singular locus of the Hurwitz complex of the dual surface to a smooth curve has just one component (namely the bitangent congruence), although we expect that the singular locus of the Hurwitz complex of a general surface has two components by Theorem 4.2.

Remark 5.7. Let now $S \subset \mathbb{P}^3$ be a surface with dual surface S^\vee . Then we have $\text{CH}_1(S)^\perp = \text{CH}_1(S^\vee)$ [19, Thm. 20, also Ex. 22]. If both S and S^\vee have mild singularities, Arrondo et. al. discuss in the proof of [2, Lemma 5.1] that $\text{Bit}(S)^\perp = \text{Bit}(S^\vee)$. This can be shown similarly to Theorem 5.5.

6 Intersection Theory

In this section we recall well-known facts about the intersection theory of $\text{Gr}(1, \mathbb{P}^3)$, and we explain the relevance of these techniques to the problems we study.

Every smooth algebraic variety X of dimension n has an associated graded ring $A^*(X)$ called the *Chow ring* of X , which is defined as follows. Let $Z^d(X)$ be the free abelian group generated by closed subvarieties of X of codimension d . An element of $Z^d(X)$ is called a *cycle*. A cycle $\alpha \in Z^d(X)$ is *principal* if there exists a closed subvariety V of codimension $(d-1)$ and a rational function f on V such that α is the divisor of f . Then the d -th Chow group $A^d(X)$ is $Z^d(X)/\sim$, where \sim denotes the subgroup generated by all principal cycles. The Chow ring $A^*(X)$ equals, as a

group, $\bigoplus_{d=0}^n A^d(X)$. Two cycles which are equivalent in the Chow ring are called *rationally equivalent*.

Since for an irreducible X the only codimension 0 closed subvariety of X is X itself, we have that $A^0(X) \simeq \mathbb{Z}$ and that rational equivalence of divisors is the same as linear equivalence. This implies that $A^1(X)$ equals $\text{Pic}(X)$, and we can consider the Chow groups $A^d(X)$ as generalizations of the Picard group of a variety.

The ring structure on $A^*(X)$ is, intuitively speaking, given by intersecting subvarieties, that is $[V][W] = [V \cap W]$. However, this is only true if the varieties V, W intersect transversally. If they do not, there are various ways to get around this problem, which we will not go into here.

Example 6.1. The Chow ring of the projective space \mathbb{P}^n is isomorphic to $\mathbb{Z}[H]/H^{n+1}$, where H is the class of a hyperplane. Thus any codimension d subvariety is rationally equivalent to a multiple of the intersection of d hyperplanes. We see that $A^*(\mathbb{P}^n)$ is isomorphic to the singular cohomology ring $H^*(\mathbb{P}^n; \mathbb{Z})$. This phenomenon fails in general, but it holds for Grassmannians.

We will now describe the Chow ring of the Grassmannian $\text{Gr}(1, \mathbb{P}^3)$ of lines in \mathbb{P}^3 . Most of this material can be found in [1] or [32]. Fix a complete flag

$$p \in l \subset h \subset \mathbb{P}^3$$

of a point p contained in a line l contained in a plane $h \subset \mathbb{P}^3$. We define the following subvarieties, which are called *Schubert varieties* and have specified intersections with respect to our chosen flag:

$$\begin{aligned} \Sigma_0 &= \text{Gr}(1, \mathbb{P}^3), \\ \Sigma_1 &= \{L \mid L \cap l \neq \emptyset\} \subset \text{Gr}(1, \mathbb{P}^3), \\ \Sigma_{1,1} &= \{L \mid L \subset h\} \subset \text{Gr}(1, \mathbb{P}^3), \\ \Sigma_2 &= \{L \mid p \in L\} \subset \text{Gr}(1, \mathbb{P}^3), \\ \Sigma_{2,1} &= \{L \mid p \in L \subset h\} \subset \text{Gr}(1, \mathbb{P}^3), \\ \Sigma_{2,2} &= \{l\} \subset \text{Gr}(1, \mathbb{P}^3). \end{aligned}$$

The rational equivalence class of a Schubert variety Σ_I is denoted by σ_I and is called a *Schubert cycle*. It is a well-known fact that these cycles form a basis for the Chow ring $A^*(\text{Gr}(1, \mathbb{P}^3))$ [8, Theorem 5.26], where the sum of the subscripts of any σ_I is the codimension of σ_I , i.e., we have

$$\begin{aligned} A^0(\text{Gr}(1, \mathbb{P}^3)) &\simeq \mathbb{Z}\sigma_0, \\ A^1(\text{Gr}(1, \mathbb{P}^3)) &\simeq \mathbb{Z}\sigma_1, \\ A^2(\text{Gr}(1, \mathbb{P}^3)) &\simeq \mathbb{Z}\sigma_{1,1} \oplus \mathbb{Z}\sigma_2, \\ A^3(\text{Gr}(1, \mathbb{P}^3)) &\simeq \mathbb{Z}\sigma_{2,1}, \\ A^4(\text{Gr}(1, \mathbb{P}^3)) &\simeq \mathbb{Z}\sigma_{2,2}. \end{aligned}$$

There is a transitive group action of $GL_{n+1}(\mathbb{C})$ on $\text{Gr}(k, \mathbb{P}^n)$. To determine the intersection products, we appeal to a result known as Kleiman's Transversality Theorem [20]. It implies that for two given subvarieties $X, Y \in \text{Gr}(k, \mathbb{P}^n)$, a general translate Z of Y under the $GL_{n+1}(\mathbb{C})$ -action is rationally equivalent to Y and the intersection of X and Z is transversal at a general point of any component of $X \cap Z$. In particular this implies $[X][Y] = [X \cap Z]$.

To determine the product $\sigma_{1,1}\sigma_2$ we can then simply intersect general varieties representing these classes: $\sigma_{1,1}$ consists of all lines L contained in a fixed plane h , while σ_2 is all lines L containing a fixed point p . Since a general point p does not lie in a general plane h , we get that $\sigma_{1,1}\sigma_2 = 0$. By similar arguments we can determine all intersection products:

$$\begin{aligned}\sigma_{1,1}^2 &= \sigma_{2,2}, \\ \sigma_2^2 &= \sigma_{2,2}, \\ \sigma_{1,1}\sigma_2 &= 0, \\ \sigma_1\sigma_{2,1} &= \sigma_{2,2}, \\ \sigma_1\sigma_{1,1} &= \sigma_{2,1}, \\ \sigma_1\sigma_2 &= \sigma_{2,1}, \\ \sigma_1^2 &= \sigma_2 + \sigma_{1,1}.\end{aligned}\tag{9}$$

To relate all this to the previous sections, we see that the degree of a subvariety of $\text{Gr}(1, \mathbb{P}^3)$ introduced in the introduction is nothing but its degree in the appropriate Chow group $A^i(\text{Gr}(1, \mathbb{P}^3))$. For instance the order α of a surface $X \subset \text{Gr}(1, \mathbb{P}^3)$ is geometrically the number of lines in X passing through a general point. This is the same as the σ_2 -coefficient of X in $A^2(\text{Gr}(1, \mathbb{P}^3))$, since we can intersect X with a general variety representing σ_2 by the Transversality Theorem. Similarly the class of X is the $\sigma_{1,1}$ -coefficient of X in $A^2(\text{Gr}(1, \mathbb{P}^3))$. In a similar way the degree of a curve or threefold in $\text{Gr}(1, \mathbb{P}^3)$ is its coefficient in $A^3(\text{Gr}(1, \mathbb{P}^3))$ or $A^1(\text{Gr}(1, \mathbb{P}^3))$, respectively.

Above we gave geometric proofs for the bidegrees of $\text{Sec}(C)$, $\text{Bit}(S)$ and $\text{Infl}(S)$. However, it is also possible to use the more technical machinery of Chern classes to prove this kind of claims. A good introduction to Chern classes in algebraic geometry is given in [32]. We will give a short exposition of the basics of this theory, and explain how it is related to the previous sections.

For any vector bundle \mathcal{E} of rank r on a variety X , one can associate *Chern classes* $c_i(\mathcal{E}) \in A^i(X)$ for $i = 0, \dots, r$. We will not give a full definition of these. However, for a globally generated vector bundle one can define them as degeneracy loci. For this, pick general global sections s_1, \dots, s_j of \mathcal{E} . The cycle of the locus $D(s_1, \dots, s_j)$ of points $x \in X$, where these fail to be linearly independent, equals the Chern class $c_{r+1-j}(\mathcal{E})$. In particular, $c_r(\mathcal{E})$ is represented by the vanishing locus of a general global section.

For a vector bundle \mathcal{E} denote by $c(\mathcal{E})$ the *total Chern class* $1 + c_1(\mathcal{E}) + \dots + c_r(\mathcal{E})$. Then we have the following [10, Theorem 3.2 (e), Remark 3.2.3 (a)]:

- (Whitney sum formula) For an exact sequence of vector bundles

$$0 \rightarrow \mathcal{E} \rightarrow \mathcal{F} \rightarrow \mathcal{G} \rightarrow 0$$

we have $c(\mathcal{F}) = c(\mathcal{E})c(\mathcal{G})$.

- $c_i(\mathcal{E}^*) = (-1)^i c_i(\mathcal{E})$, where \mathcal{E}^* is the dual vector bundle to \mathcal{E} .

Example 6.2. Consider the vector bundle $\mathcal{E} = \mathcal{O}(a_1) \oplus \mathcal{O}(a_2) \oplus \cdots \oplus \mathcal{O}(a_n)$ on \mathbb{P}^n where all $a_i > 0$. Since $\mathcal{O}(a_i)$ is globally generated, $c_1(\mathcal{O}(a_i))$ is the locus where a general global section vanishes, i.e., the locus where a general homogeneous polynomial of degree a_i in $n+1$ variables vanishes. The class of this in the Chow ring of \mathbb{P}^n is $a_i H$. By iterated use of the Whitney sum formula we have $c_n(\mathcal{E}) = \prod_{i=1}^n c_1(\mathcal{O}(a_i)) = \prod_{i=1}^n a_i H^n$.

On the one hand, $c_n(\mathcal{E})$ is the locus where a general section vanishes. In other words it is the common zeros of general polynomials of degree a_1, \dots, a_n . On the other hand, it is represented by $\prod_{i=1}^n a_i$ times the point H^n . Thus we have recovered Bezout's Theorem in the case of general polynomials.

Example 6.3. We write $T_{\mathbb{P}^n}$ for the tangent bundle on \mathbb{P}^n . We have the well known Euler exact sequence

$$0 \rightarrow \mathcal{O}_{\mathbb{P}^n} \rightarrow \mathcal{O}_{\mathbb{P}^n}(1)^{\oplus(n+1)} \rightarrow T_{\mathbb{P}^n} \rightarrow 0.$$

By the Whitney sum formula we have $c(T_{\mathbb{P}^n}) = (1 + c_1(\mathcal{O}_{\mathbb{P}^n}(1)))^{n+1} = (1 + H)^{n+1}$, where H is the class of a hyperplane. Thus for instance $c_1(T_{\mathbb{P}^n}) = (n+1)H$.

Example 6.4. Building on the previous example, let $X \subset \mathbb{P}^n$ be a smooth hypersurface of degree d . We will compute the Chern classes of the tangent bundle T_X . We have the exact sequence

$$0 \rightarrow T_X \rightarrow T_{\mathbb{P}^n}|_X \rightarrow N_{X|\mathbb{P}^n} \rightarrow 0,$$

where $N_{X|\mathbb{P}^n}$ is the normal bundle of X in \mathbb{P}^n .

It is well-known that in this case $N_{X|\mathbb{P}^n} = \mathcal{O}_{\mathbb{P}^n}(d)|_X$. Applying the Whitney sum formula we get $c(T_X)c(\mathcal{O}_{\mathbb{P}^n}(d)|_X) = c(T_{\mathbb{P}^n}|_X)$. Letting $h = H|_X$ we get $c(T_X)(1 + dh) = (1 + h)^{n+1}$, or written differently $c(T_X) = \frac{(1+h)^{n+1}}{1+dh} = (1+h)^{n+1}(1 - dh + d^2h^2 - \cdots)$, where the last equality is obtained by formally expanding the power series. From this we can determine individual Chern classes, for instance we have that $c_1(T_X) = (n+1)h - dh = h(n+1-d)$ and $c_2(T_X) = \binom{n+1}{2}h^2 - (n+1)dh^2 + d^2h^2 = h^2(\frac{n(n+1)}{2} - (n+1)d + d^2)$.

On $\text{Gr}(1, \mathbb{P}^3)$ there is the tautological exact sequence of vector bundles

$$0 \rightarrow S \rightarrow \mathcal{O}_{\text{Gr}(1, \mathbb{P}^3)}^{\oplus 4} \rightarrow Q \rightarrow 0,$$

where S is the tautological rank 2 subbundle whose fiber at a point $W \in \text{Gr}(1, \mathbb{P}^3)$ (now considered as a 2-dimensional subvectorspace of \mathbb{C}^4) is W itself, and Q is the

quotient whose fiber at W is \mathbb{C}^4/W . Both S^* and Q are globally generated and we have that $H^0(\mathrm{Gr}(1, \mathbb{P}^3), S^*) = (\mathbb{C}^4)^*$ and $H^0(\mathrm{Gr}(1, \mathbb{P}^3), Q) = \mathbb{C}^4$ (see [1, Prop 0.5]).

Consider a general element $\phi \in (\mathbb{C}^4)^*$, i.e., a general linear map $\phi : \mathbb{C}^4 \rightarrow \mathbb{C}$ (in this case it is sufficient to require that $\phi \neq 0$). This is a section of S^* whose value at the point $[W]$ is $\phi|_W : W \rightarrow \mathbb{C}$. According to the degeneracy locus definition, $c_2(S^*)$ is the locus where this vanishes. This is equivalent to $W \subset \ker \phi$, and by the generality assumption we know that $\ker \phi$ is a hyperplane. In other words $c_2(S^*) = \sigma_{1,1}$, and hence also $c_2(S) = \sigma_{1,1}$.

If we instead consider two general sections $\phi, \psi : \mathbb{C}^4 \rightarrow \mathbb{C}$ of S^* , then $c_1(S^*)$ is the locus of points $[W]$ where $\phi|_W$ and $\psi|_W$ fail to be linearly independent. This is exactly the condition that $W \cap (\ker \phi \cap \ker \psi) \neq \{0\}$. By generality $(\ker \phi \cap \ker \psi)$ is a 2-dimensional subspace of \mathbb{C}^4 , and thus $c_1(S^*) = \sigma_1$.

For Q we can give a similar argument. A global section is given by the choice of a point $p \in \mathbb{C}^4$. Its value at $[W]$ is simply the image of p in \mathbb{C}^4/W . Thus $c_2(Q)$ is the locus of points $[W]$ containing the point p , which equals σ_2 . Similarly the condition that two general sections p, q are linearly dependent at $[W]$ means that the 2-dimensional subspace of \mathbb{C}^4 spanned by p and q intersects W non-trivially, i.e., $c_1(Q) = \sigma_1$.

Now we have all the machinery one needs to explain the bidegree of a congruence in terms of Chern classes. Consider a surface $X \subset \mathrm{Gr}(1, \mathbb{P}^3)$. Its class in the Chow group $A^2(\mathrm{Gr}(1, \mathbb{P}^3))$ is a linear combination $[X] = \alpha\sigma_2 + \beta\sigma_{1,1}$. We have

$$c_2(Q)[X] = \sigma_2(\alpha\sigma_2 + \beta\sigma_{1,1}) = \alpha\sigma_{2,2},$$

and by the same argument $c_2(S)[X] = \beta\sigma_{2,2}$. Hence determining the bidegree is equivalent to determining $(c_2(Q)[X], c_2(S)[X])$.

This may seem like a lot of complicated work to only reformulate the question, but in many cases it might be easier to calculate these things using all the formal properties of Chern classes. This is the way one would try to compute bidegrees for most congruences, since finding geometric arguments might be difficult.

For a variety $X \subset \mathbb{P}^n$, the degrees of the coisotropic hypersurfaces $\mathrm{CH}_i(X)$ are expressible in terms of Chern classes.

Example 6.5. Let S be a smooth surface in \mathbb{P}^3 . We determined the degree of the Hurwitz complex $\mathrm{CH}_1(S)$ in Corollary 5.3. Now we explain a second way to compute the degree of $\mathrm{CH}_1(S)$, which is its coefficient in $A^1(\mathrm{Gr}(1, \mathbb{P}^3))$. By [19, Theorem 9] this degree is equal to the degree $\delta_1(S)$ of the first polar locus $M_1(S) = \{x \in S \mid y \in T_x S\}$, where y is a general point of \mathbb{P}^3 (this locus is the polar curve in the proof of Theorem 4.3). Letting T_S be the tangent bundle of S , we have by [10, Example 14.4.15] that

$$\delta_1(S) = \deg(3h - c_1(T_S)).$$

By Example 6.4 we get $\delta_1(S) = \deg(3h - h(3 + 1 - d)) = (d - 1)\deg(h)$. Since S is a degree d surface, the degree of the hyperplane h equals d . Thus $\delta_1(S) = d(d - 1)$.

6.1 Applications

The discussion on intersection theory allows us to quickly perform computations which are otherwise very tricky. We now give two examples of this.

Example 6.6 (Problem 3 on Grassmannians in [31]). Let $S_1, S_2 \subset \mathbb{P}^3$ be general surfaces of degree d_1 and d_2 , respectively, where $d_1, d_2 \geq 4$. How many lines are bitangent to both surfaces?

To answer this question we need to compute the cardinality of the set $\text{Bit}(S_1) \cap \text{Bit}(S_2)$. As the surfaces were chosen generally, this computation can be performed with the tools of intersection theory. In particular we know from Theorem 4.3 and from Section 6 that $[\text{Bit}(S_i)] = \alpha_i \sigma_2 + \beta_i \sigma_{1,1}$, where

$$\alpha_i = \frac{1}{2} d_i (d_i - 1) (d_i - 2) (d_i - 3),$$

$$\beta_i = \frac{1}{2} d_i (d_i - 2) (d_i - 3) (d_i + 3),$$

and $i \in \{1, 2\}$. Using the formulas in (9) we get that $[\text{Bit}(S_1)][\text{Bit}(S_2)] = (\alpha_1 \alpha_2 + \beta_1 \beta_2) \sigma_{2,2}$. Hence, the required number is equal to

$$\begin{aligned} & \frac{1}{4} d_1 (d_1 - 1) (d_1 - 2) (d_1 - 3) d_2 (d_2 - 1) (d_2 - 2) (d_2 - 3) \\ & + \frac{1}{4} d_1 (d_1 - 2) (d_1 - 3) (d_1 + 3) d_2 (d_2 - 2) (d_2 - 3) (d_2 + 3). \end{aligned}$$

Example 6.7. Let $S \subset \mathbb{P}^3$ be a general surface of degree $d_1 \geq 4$, and let $C \subset \mathbb{P}^3$ be a general curve of degree $d_2 \geq 2$ and geometric genus g . How many lines are bitangent to S and secant to C ?

The answer is again a simple computation in the Chow ring of the Grassmannian. We need to multiply the classes

$$\begin{aligned} [\text{Bit}(S)] &= \frac{1}{2} d_1 (d_1 - 1) (d_1 - 2) (d_1 - 3) \sigma_2 + \frac{1}{2} d_1 (d_1 - 2) (d_1 - 3) (d_1 + 3) \sigma_{1,1}, \\ [\text{Sec}(C)] &= \left(\frac{1}{2} (d_2 - 1) (d_2 - 2) - g \right) \sigma_2 + \frac{1}{2} d_2 (d_2 - 1) \sigma_{1,1}. \end{aligned}$$

This gives us that the required number is

$$\begin{aligned} & \frac{1}{4} d_1 (d_1 - 1) (d_1 - 2) (d_1 - 3) ((d_2 - 1) (d_2 - 2) - 2g) \\ & + \frac{1}{4} d_1 (d_1 - 2) (d_1 - 3) (d_1 + 3) d_2 (d_2 - 1). \end{aligned}$$

7 Singular Loci of Congruences

In this section we investigate the singular points of the congruences we have introduced. We begin by studying the singularities of the secant locus $\text{Sec}(C)$ for a smooth irreducible curve $C \subset \mathbb{P}^3$ that is not contained in any plane. Note that the secant congruence of a curve that is not a line but contained in a plane is simply the set of all lines lying in the same plane; hence this congruence is smooth.

Proposition 7.1. *Let L be a line that intersects C in three or more distinct points. Then L represents a singular point of $\text{Sec}(C)$.*

Proof. The symmetric square $C^{(2)}$ of the curve C is the quotient of $C \times C$ by the action of the symmetric group S_2 . In other words, $C^{(2)}$ is the variety whose points are unordered pairs of points on C . It is again a projective variety [14, pp. 126–127]. For two distinct points $x, y \in C$ we denote by $\langle x, y \rangle$ the line spanned by x and y . With this, we have the following birational morphism between the symmetric square of C and the secant congruence of C :

$$\begin{aligned} \pi : C^{(2)} &\longrightarrow \text{Sec}(C), \\ \{x, y\} &\longmapsto \begin{cases} \langle x, y \rangle, & \text{if } x \neq y \\ T_x C, & \text{if } x = y \end{cases}. \end{aligned}$$

The fiber $\pi^{-1}(L)$ is a finite set containing more than one element since $|L \cap C| \geq 3$. Hence, $\pi^{-1}(L)$ is not connected and the secant congruence $\text{Sec}(C)$ is singular at L by Theorem 3.4. \square

We use the following result without proof.

Proposition 7.2 ([2, Lemma 2.3]). *Let L be a line that intersects C in exactly two points x, y . Then L represents a smooth point of $\text{Sec}(C)$ if and only if L is different from $T_x C$ and $T_y C$.*

Finally we have to consider lines in $\text{Sec}(C)$ that meet the curve in a single point. For this, we need the following technical lemma.

Lemma 7.3. *Let $f \in \mathbb{C}[[z, w]]$ be a power series in two variables. Suppose f is skew-symmetric, i.e., $f(z, w) = -f(w, z)$. Then $z - w$ divides f .*

Proof. We write f as a sum of homogeneous polynomials $f = \sum_i f_i$. As $f(z, w) + f(w, z) = 0$ it follows that, in each degree i , we have $f_i(z, w) + f_i(w, z) = 0$, so that each f_i is skew-symmetric. In particular we get that $f_i(w, w) = 0$. If we consider $f_i(w, z)$ as a polynomial in the variable z and coefficients in $\mathbb{C}[w]$, the last equality means that w is a root of f_i . By the factor theorem for univariate polynomials, we have that $z - w$ divides f_i for every i , and this completes the proof. \square

Theorem 7.4. *Let $L \in \text{Sec}(C)$ be a line that intersects C in a single point x . Then the intersection multiplicity of L and C at x is at least two. Moreover the line L represents a smooth point of $\text{Sec}(C)$ if and only if the multiplicity is exactly two.*

Proof. Suppose L intersects C in x with multiplicity two.

We thank Jenia Tevelev for suggesting the argument of this part of the proof.

As we are dealing with a local statement, we can work in the affine open set where $x_3 \neq 0$. Without loss of generality we assume $x = (0 : 0 : 0 : 1)$, and $L = V(x_1, x_2)$. In our affine coordinates we have in particular $x = (0, 0, 0)$.

We now consider C as an analytic space. As C is smooth, there is a local analytic isomorphism φ between \mathbb{A}^1 around the origin and C around the point x . The map φ will have the form $\varphi(z) = (t_0(z), t_1(z), t_2(z))$ for $t_0, t_1, t_2 \in \mathbb{C}[[z]]$. As L is the tangent line of C at x , we have that $t'_0(0) \neq 0$, while $t'_1(0) = t'_2(0) = 0$. Therefore we can apply an analytic change of coordinates so that $\varphi(z) = (z, t_1(z), t_2(z))$. Moreover, as L is a simple tangent, at least one of t_1 and t_2 must vanish at 0 with order exactly two: we will assume it is t_1 , so $t'_1(0) \neq 0$. We can thus write $t_1 = z^2 + z^3 f(z)$ and $t_2 = z^2 g(z)$ for some $f, g \in \mathbb{C}[[z]]$.

The line spanned by two points $\varphi(z)$ and $\varphi(w)$ of C is the row space of a matrix of the form

$$\begin{pmatrix} z & z^2 + z^3 f(z) & z^2 g(z) & 1 \\ w & w^2 + w^3 f(w) & w^2 g(w) & 1 \end{pmatrix}.$$

All the Plücker coordinates are skew-symmetric power series in z and w , and are thus divisible by $z - w$ according to Lemma 7.3. If we set $f(z) = \sum a_i z^i$ and $g(z) = \sum b_i z^i$, we have in particular

$$\begin{aligned} p_{01} &= z(w^2 + w^3 f(w)) - w(z^2 + z^3 f(z)) = zw(w - z + \sum_i a_i (w^{i+2} - z^{i+2})) \\ &= zw(w - z)(1 + \sum_i a_i \sum_{j=0}^{i+1} w^j z^{i-j+1}), \\ p_{03} &= z - w, \\ p_{13} &= z^2 + z^3 f(z) - w^2 - w^3 f(w) = z^2 - w^2 + \sum_i a_i (z^{i+3} - w^{i+3}) \\ &= (z - w)(z + w + \sum_i a_i \sum_{j=0}^{i+2} z^j w^{i+2-j}). \end{aligned}$$

By homogeneity we can divide all of them by $z - w$. We recall that the symmetric product $(\mathbb{A}^1)^{(2)}$ of the affine line \mathbb{A}^1 with itself is a smooth surface isomorphic to the affine plane \mathbb{A}^2 (see [14, p. 126]). Consider the morphism $(\mathbb{A}^1)^{(2)} \rightarrow \text{Sec}(C)$ that sends $\{z, w\}$ to the line spanned by $\varphi(z)$ and $\varphi(w)$ if $z \neq w$ and that sends $\{z, z\}$ to the tangent line of C at z . We will show that $\text{Sec}(C)$ is smooth at L by showing that this is a local isomorphism. The above computations show that this morphism is described as

$$\{z, w\} \mapsto (p_{01} = -zw + h_1 : p_{02} : p_{03} = 1 : p_{12} : p_{13} = z + w + h_2 : p_{23}),$$

where all monomials of h_1 have degree at least three and all monomials of h_2 have degree at least two. The forms zw and $z + w$ are local coordinates of $(\mathbb{A}^1)^{(2)} = \text{Spec}(\mathbb{C}[zw, z + w])$ around the origin, so that the morphism is a local isomorphism as claimed.

Now suppose that the intersection multiplicity of L and C at x is at least three. Then the line L is contained in the closure of the set of lines that intersect C in at least three points or that intersect C in two points one of which with multiplicity at least two. By Propositions 7.1 and 7.2, the line is singular in $\text{Sec}(C)$. \square

We want to derive an analogous description for the singularities of the bitangent locus $\text{Bit}(S)$ and the inflectional locus $\text{Infl}(S)$ of a surface $S \subset \mathbb{P}^3$.

Theorem 7.5. *Let $S \subset \mathbb{P}^3$ be an irreducible smooth surface of degree at least four which does not contain any lines. Then the singular locus of $\text{Infl}(S)$ consists of lines which*

- *intersect S with multiplicity at least three at two or more distinct points, or*
- *intersect S with multiplicity at least four at some point.*

Proof. We consider the incidence variety

$$A_S := \overline{\{(x, L) \mid L \text{ intersects } S \text{ at } x \text{ with multiplicity } 3\}} \subset S \times \text{Gr}(1, \mathbb{P}^3).$$

The projection

$$\begin{aligned} \pi : A_S &\longrightarrow \text{Infl}(S), \\ (x, L) &\longmapsto L \end{aligned}$$

is a surjective morphism. Since S does not contain any lines, all fibers of π are finite and thus π is finite by Lemma 3.5. Moreover the general fiber of π has cardinality one, so π is birational. Hence, we can apply Proposition 3.2 to the projection π . For this, we need to examine the singularities of A_S and the differentials of π .

We denote the defining equation for S by f and consider the affine chart where $x_0 \neq 0$ and the Plücker coordinate $p_{01} \neq 0$. Then we may assume $x = (1 : \alpha : \beta : \gamma)$ and L is spanned by $(1 : 0 : a : b)$ and $(0 : 1 : c : d)$. In this affine chart, S is defined by $g_0(\alpha, \beta, \gamma) := f(1, \alpha, \beta, \gamma)$. As in the proof of Theorem 3.7, we have that $x \in L$ if and only if $a = \beta - \alpha c$ and $b = \gamma - \alpha d$. Parametrizing the line L by $\ell(u, v) := (u : u\alpha + v : u\beta + vc : u\gamma + vd)$ for $(u : v) \in \mathbb{P}^1$ shows that L intersects S with multiplicity at least m at x if and only if $f(\ell(u, v))$ is divisible by v^m . This is equivalent to that $\frac{\partial f(\ell(u, v))}{\partial v} = \dots = \frac{\partial^{m-1} f(\ell(u, v))}{\partial v^{m-1}} = 0$ whenever $v = 0$. This means exactly that

$$g_k := \left(\frac{\partial}{\partial \alpha} + c \frac{\partial}{\partial \beta} + d \frac{\partial}{\partial \gamma} \right)^k g_0 = 0 \text{ for all } k = 1, \dots, m-1. \quad (10)$$

Hence, in the described affine chart, A_S can be written as

$$\{(\alpha, \beta, \gamma, a, b, c, d) \mid g_0 = 0, a = \beta - \alpha c, b = \gamma - \alpha d, g_1 = 0, g_2 = 0\}$$

and its Jacobian matrix has the form

$$\begin{pmatrix} \frac{\partial g_0}{\partial \alpha} & \frac{\partial g_0}{\partial \beta} & \frac{\partial g_0}{\partial \gamma} & 0 & 0 & 0 & 0 \\ -c & 1 & 0 & -1 & 0 & -\alpha & 0 \\ -d & 0 & 1 & 0 & -1 & 0 & -\alpha \\ \frac{\partial g_1}{\partial \alpha} & \frac{\partial g_1}{\partial \beta} & \frac{\partial g_1}{\partial \gamma} & 0 & 0 & \frac{\partial g_0}{\partial \beta} & \frac{\partial g_0}{\partial \gamma} \\ \frac{\partial g_2}{\partial \alpha} & \frac{\partial g_2}{\partial \beta} & \frac{\partial g_2}{\partial \gamma} & 0 & 0 & 2\frac{\partial g_1}{\partial \beta} & 2\frac{\partial g_1}{\partial \gamma} \end{pmatrix}. \quad (11)$$

Claim. The differential $d_{(x,L)}\pi : T_{(x,L)}A_S \rightarrow T_L\text{Infl}(S)$ is not injective if and only if the intersection multiplicity of L and S at x is at least four.

Proof. The differential $d_{(x,L)}\pi$ projects every element in the kernel of (11) on its last four coordinates. This is not an injection if and only if there is an element of the form $(\tau, \sigma, \rho, 0, 0, 0, 0)^T \neq 0$ in the kernel of (11). The second and third row of (11) imply that $\sigma = c\tau$ and $\rho = d\tau$. Hence, the first, fourth and fifth row of (11) yield $g_1(\alpha, \beta, \gamma) = g_2(\alpha, \beta, \gamma) = g_3(\alpha, \beta, \gamma) = 0$. Now the claim follows from (10). \diamond

Claim. If A_S is singular at (x, L) , then the intersection multiplicity of L and S at x is at least four.

Proof. The incidence variety A_S is singular at (x, L) if and only if the matrix (11) does not have full rank. Since S is smooth, the first four rows of this matrix are linearly independent. If A_S is singular at (x, L) , the fifth row of (11) can be written as a linear combination of the other four rows. Actually it has to be a linear combination of the first and the fourth row. This shows that $\frac{\partial g_2}{\partial \alpha} = \mu \frac{\partial g_0}{\partial \alpha} + \nu \frac{\partial g_1}{\partial \alpha}$, $\frac{\partial g_2}{\partial \beta} = \mu \frac{\partial g_0}{\partial \beta} + \nu \frac{\partial g_1}{\partial \beta}$ and $\frac{\partial g_2}{\partial \gamma} = \mu \frac{\partial g_0}{\partial \gamma} + \nu \frac{\partial g_1}{\partial \gamma}$ for some $\mu, \nu \in \mathbb{C}$. Hence, $g_3(\alpha, \beta, \gamma) = \mu g_1(\alpha, \beta, \gamma) + \nu g_2(\alpha, \beta, \gamma) = 0$ and L intersects S at x with multiplicity at least four by (10). \diamond

Since the fiber $\pi^{-1}(L)$ consists of more than one point if and only if L intersects S with multiplicity at least three at two or more distinct points, Proposition 3.2 yields Theorem 7.5. \square

To study the singular locus of the bitangent congruence $\text{Bit}(S)$, we assume that $S \subset \mathbb{P}^3$ is a general irreducible surface of degree at least four.

Proposition 7.6. *Let L be a line that is tangent to S at three or more distinct points. Then L represents a singular point of $\text{Bit}(S)$.*

Proof. As in the proof of Proposition 7.1, we consider the symmetric square $S^{(2)}$, which is the quotient of $S \times S$ by the action of S_2 . This is again a projective variety [14, pp. 126–127]. The projection π from

$$\overline{\{(\{x, y\}, L) \mid x \neq y, x, y \in L \subset T_x S \cap T_y S\}} \subset S^{(2)} \times \text{Gr}(1, \mathbb{P}^3)$$

onto $\text{Bit}(S)$ defined by $(\{x, y\}, L) \mapsto L$ is a birational morphism. The fiber $\pi^{-1}(L)$ is a finite set containing more than one element since L is tangent to S in at least three

distinct points. Hence, $\pi^{-1}(L)$ is not connected and the line L is a singular point of $\text{Sec}(C)$ by Theorem 3.4. \square

We use another result without proof.

Proposition 7.7 ([2, Lemma 4.3]).

- *Let L be a line that is tangent to S at exactly two points x, y . Then L represents a smooth point of $\text{Bit}(S)$ if and only if the intersection multiplicity of L and S at both x and y is exactly two.*
- *Let $L \in \text{Bit}(S)$ be a line that is tangent to S at a single point x . Then the intersection multiplicity of L and S at x is at least four. Moreover, the line L represents a smooth point of $\text{Bit}(S)$ if the multiplicity is exactly four.*

Unfortunately we do not yet have a full understanding of lines in $\text{Bit}(S)$ that have an intersection multiplicity greater than four at a point of S .

Conjecture 7.8. Let $L \in \text{Bit}(S)$ be a line that is tangent to S at a single point x such that the intersection multiplicity of L and S at x is at least five. Then L represents a singular point of $\text{Bit}(S)$.

Even for a surface S of degree five we do not know how to prove this. The reason for this is the following. There are in principle two possibilities on how to approximate the line L by lines in $\text{Bit}(S)$ with lower intersection multiplicities. Either L is in the closure of a set of lines that have a double and a triple point in the intersection with S (these are singular in $\text{Bit}(S)$ by Proposition 7.7), or L is in the closure of a set of lines that have two double and one single point in the intersection with S . The problem is that the last type of lines is smooth in $\text{Bit}(S)$ by Proposition 7.7, and thus we cannot conclude that L is singular.

8 Computations

As additional resources to this article we prepared some *Macaulay2* [12] code to compute the varieties we have discussed.

To compute the Chow form of a space curve C , we first compute the incidence variety A_C and then the image of the projection in (3). More explicitly, we add the equations describing a line ($a_0x_0 + a_1x_1 + a_2x_2 + a_3x_3 = 0$ and $b_0x_0 + b_1x_1 + b_2x_2 + b_3x_3 = 0$) to the ideal $I \subset \mathbb{C}[x_0, x_1, x_2, x_3]$ defining the curve C . To get a correct description of the incidence variety we need to saturate by the irrelevant ideal generated by x_0, x_1, x_2, x_3 since every projective line passes through the affine origin $(0, 0, 0, 0)$. The projection in (3) is realized by eliminating the variables x_0, x_1, x_2, x_3 . After this we are left with an ideal in the variables $a_0, a_1, a_2, a_3, b_0, b_1, b_2, b_3$, which is actually a hypersurface and thus generated by a single SL_2 -invariant polynomial. Using [29, Algorithm 3.2.8], we can transform this polynomial to a polynomial in the Plücker variables.

In *Macaulay2* code this looks as follows:

```

R=QQ[x_0..x_3,a_0..a_3,b_0..b_3];
Inc=sub(I,R)+ideal(a_0*x_0+a_1*x_1+a_2*x_2+a_3*x_3,
b_0*x_0+b_1*x_1+b_2*x_2+b_3*x_3);
Inc=saturate(Inc,ideal(x_0,x_1,x_2,x_3));
CH=(mingens eliminate({x_0,x_1,x_2,x_3},Inc))_(0,0);
toPluecker CH

```

The command `toPluecker` is our implementation of [29, Algorithm 3.2.8].

Now we have two possibilities to compute the secant congruence of C : either we compute the singular locus of the Chow complex and identify the secant congruence among the irreducible components of this singular locus (cf. Thm. 3.7), or we compute the secant congruence directly using the same ideas as above. For the latter approach, we compute the incidence variety $\{(x,y,L) \mid x,y \in C, L \in \text{Gr}(1, \mathbb{P}^3), x,y \in L, x \neq y\}$ by having the equations in I and the equations for L in the variables x_0, x_1, x_2, x_3 and y_0, y_1, y_2, y_3 . To ensure $x \neq y$, we need to saturate by the 2×2 -minors of the matrix $\begin{pmatrix} x_0 & x_1 & x_2 & x_3 \\ y_0 & y_1 & y_2 & y_3 \end{pmatrix}$. Finally, we eliminate the variables $x_0, x_1, x_2, x_3, y_0, y_1, y_2, y_3$.

The above algorithm for Chow forms can be adjusted to compute the Hurwitz form of a surface $S \subset \mathbb{P}^3$. Let S be defined by the polynomial $f \in \mathbb{C}[x_0, x_1, x_2, x_3]$, and let I be the ideal generated by f . We first compute the incidence variety of all $(x, L) \in \mathbb{P}^3 \times \text{Gr}(1, \mathbb{P}^3)$ such that L is tangent to S at x . This is done by computing the ideal `Inc` as above and by adding to it the condition that L is contained in the tangent plane $T_x S$. This last condition is simply described by the vanishing of the 3×3 -minors of the matrix $\begin{pmatrix} a_0 & a_1 & a_2 & a_3 \\ b_0 & b_1 & b_2 & b_3 \\ \partial_{x_0} f & \partial_{x_1} f & \partial_{x_2} f & \partial_{x_3} f \end{pmatrix}$. Then we saturate by the singular locus of S . The saturation is needed, otherwise we would not only get the Hurwitz complex, but also extra components corresponding to the set of lines meeting the singular locus of S . If the surface S is smooth, one can reduce this step by saturating only by the irrelevant ideal as above. Lastly we eliminate the variables x_0, x_1, x_2, x_3 to compute the image of the projection $(x, L) \mapsto L$.

The algorithm just described for the Hurwitz form reaches computational limits. Even for the smooth Fermat quartic $f = x_0^4 + x_1^4 + x_2^4 + x_3^4$ the saturation needs too much memory to terminate. The algorithm does terminate for the Fermat cubic (see [19, Examples 2 and 7]) and we can compute its bitangent congruence with the same strategy. However the bitangent congruence, and thus the singular locus of the Hurwitz complex, are most interesting for surfaces of degree at least four.

Therefore, we use the following different approach for Hurwitz forms of *smooth* surfaces. We consider a line L to be spanned by two points $(a_0 : a_1 : a_2 : a_3)$ and $(b_0 : b_1 : b_2 : b_3)$. Then we compute the intersection points of S and L by substituting every variable x_i in f with $a_i u + b_i v$. This yields a polynomial $F \in \mathbb{C}[a, b][u, v]$. The line L is tangent to S at a point parametrized by $(u : v)$ on L if and only if the three polynomials $F, \partial_u F, \partial_v F$ vanish at (u, v) . Hence, the Hurwitz form of S is the *discriminant* of F . To speed up the computations, we can also work with an affine chart of the Grassmannian by setting $a_0 = b_1 = 1$ and $a_1 = b_0 = 0$. For singular surfaces S , the discriminant of F vanishes if and only if the line L is tangent to S at a smooth point or if L meets the singular locus of S . Therefore, the discriminant

factors into the Hurwitz form of S , the Chow forms of all irreducible curves in $\text{Sing}(S)$, and the congruences of lines passing through an isolated singularity of S . Thus, one can identify the Hurwitz form of a singular surface S by inspecting the factors of the discriminant of F .

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